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Partial waves and large N_C resonance sum rules

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Abstract: Using $1/N_C$ expansion and dispersion theory techniques, without relying on any explicit resonance lagrangian, we generalize the KSRF relation by including the scalar meson effects, at leading order of chiral expansion. Two sum rules for the low energy constants L_2, L_3 and a new relation between resonance couplings are also derived. A rather detailed examination to the new relation is also given. We also discussed the N_c properties of partial wave amplitudes and the broad σ resonance.

Keywords: Sum Rules, 1/N Expansion, Chiral Lagrangians.

Contents

1. Introduction

Low energy effective field theories (EFT) are useful tools in modern particle physics [1]. The EFT lagrangian can be obtained through the integration of the heavy degrees of freedom of the whole theory. The more interesting and difficult problem is how to understand "high energy physics" from the low energy theory. In hadron physics, the low energy effective theory is chiral perturbation theory (χPT) whose degrees of freedom are just the light pseudo-Goldstone bosons from the spontaneous chiral symmetry breaking [2−4]. A former paper was devoted to the study of the inverse problem in hadron physics using techniques from S-matrix theory, low energy effective theory and $1/N_C$ expansion and it was demonstrated that resonances with $M\,,\Gamma \sim O(N_C^0)$ [5] could not exist. However, the crossed channel resonance exchange contribution to the left-hand cut were not considered in that paper. The present work performs a large-N_C calculation of the $\pi\pi$ scattering including right- and left-hand cut contributions. The analysis is taken up to next-to-leading order in the chiral expansion. This yields a consistent set of relations between the chiral couplings related to $\pi\pi$-scattering and the resonance parameters.

The partial wave amplitudes are extracted in section 2 through dispersive relations. We perform a low-energy matching to χPT in section 3. A generalized KSRF relation is extracted together with predictions for the low energy constants (LECs) L_2 and L_3. Section 4 studies the consistency of different phenomenological lagrangians under the generalized KSRF constraint. The influence of a broad sigma meson, generated through K-matrix unitarization of the current algebra amplitude, is analyzed in section 5. The results are discussed and summarized in section 6.

2. Dispersive calculation of the S-matrix

The S-matrix describing the partial wave elastic $\pi\pi$-scattering accepts the general factorization [6]

$$S \,=\, S^{\rm cut}\cdot \prod_{\rm R} S^{\rm R}\,, \tag{2.1}$$

where $S^{\rm R}$ are the simplest S-matrices characterizing isolated singularities on the second Riemann-sheet that are solutions of the generalized single-channel unitarity relations [7]. It is noticed that the eq. (2.1) is formally rigorous and can be obtained under the same condition from which the standard partial wave dispersion is derived. I.e., the so called maximal analyticity assumption or Mandelstam representation.

2.1 Contribution from the s-channel poles

2.1.1 Resonances in the s-channel

The part of the S-matrix that contains the pole singularities related to second sheet resonances is given by

$$\prod_{\rm R} S^{\rm R}(s) \,=\, \prod_{\rm R}\left(1\,+\,2\,i\,\rho(s)\,T^{\rm sR}(s)\right)\,, \tag{2.2}$$

with

$$T^{\rm sR}(s) = \frac{s\,G_{\rm R}[z_0]}{M_{\rm R}^2[z_0]-s-i\rho(s)s\,G_{\rm R}[z_0]}\,, \tag{2.3}$$

where $M_{\rm R}^2[z_0]$ and $G_{\rm R}[z_0]$ are related to the pole position $z_0\equiv\left(M+\frac{i}{2}\Gamma\right)^2$ of the resonance R [6],

$$\begin{aligned} M_{\rm R}^2[z_0] &= {\rm Re}[z_0]+\frac{{\rm Im}[z_0]\,{\rm Im}[z_0\,\rho(z_0)]}{{\rm Re}[z_0\,\rho(z_0)]}\,,\\ G_{\rm R}[z_0] &= \frac{{\rm Im}[z_0]}{{\rm Re}[z_0\,\rho(z_0)]}\,. \end{aligned} \tag{2.4}$$

The S-matrix phase-space factor is defined as $\rho(s)=\sqrt{1-4m_\pi^2/s}$, such that for $s>4m_\pi^2$ one has the prescription $\rho(s\pm i\epsilon)=\pm|\rho(s)|$. In the paper, we will refer to $\rho(s)$ as $\rho(s+i\epsilon)$. Notice that real analyticity requires the existence of a companion pole at z_0^*.

When discussing large N_C dynamics, it is not clear whether, in addition to the narrow width states lying near the physical region, there are any other S matrix poles with odd behavior. Nevertheless, the quantity $G_{\rm R}[z_0]/(M_{\rm R}^2[z_0]-4m_\pi^2)$ is always positive definite for any location of the pole z_0 in the complex s-plane [5]. Because of this, there can be no S matrix poles located on the s-plane when $N_C\to\infty$, except on the real axis or at infinity [5]. In most of this paper, we assume that all S-matrix poles indeed move to the real axis when $N_C\to\infty$. Only in section 5 we will pay some attention to the possibility that there exists a pole moving to infinity.

The s-channel second sheet resonance contribution to the T-matrix is,

$$T^{\rm sR}(s) \;=\; G_{\rm R}[z_0]\frac{s}{M_{\rm R}^2[z_0]-s} \;+\; \mathcal{O}\left(\frac{1}{N_C^2}\right)\,, \tag{2.5}$$

with the resonance parameters given in the large-N_C limit by

$$\begin{aligned} M_{\rm R}^2[z_0] &= M_{\rm R}^2\,,\\ G_{\rm R}[z_0] &= \frac{1}{\rho(M_{\rm R}^2)}\,\frac{\Gamma_{\rm R}}{M_{\rm R}}\,, \end{aligned} \tag{2.6}$$

where $M_{\rm R}$ and $\Gamma_{\rm R}$ are defined as the large N_C limit of the z_0 pole parameters M and Γ, respectively.

Eq. (2.3) would be modified in the case of resonances lying beyond the elastic region on higher Riemann sheets. However, eqs. (2.5) and (2.6) are still valid in the large-N_C limit if one replaces the width $\Gamma_{\rm R}$ by the partial decay width $\Gamma_{R\to\pi\pi}$.

The imaginary part of $T^{\rm sR}$ in eq. (2.5) shows the standard narrow-width expression

$${\rm Im}T^{\rm sR}(s) \;=\; \pi\,\frac{M_{\rm R}\,\Gamma_{\rm R}}{\rho(M_{\rm R}^2)}\;\delta(s-M_{\rm R}^2)\,. \tag{2.7}$$

This expression can be directly extracted from the imaginary part of eq. (2.3) in the limit $G_{\rm R}[z_0]\to 0$. Eq. (2.5) is recovered back through a once-subtracted T-matrix dispersion relation.

The expansion of $\prod\limits_{\rm R} S^{\rm R}$ in $1/N_C$ is given at the first non-trivial order by

$$\prod_{\rm R} S^{\rm R}(s) \;=\; 1 \;+\; 2\,i\,\rho(s)\,\sum_{R} T^{\rm sR}(s) \;+\; \mathcal{O}\left(\frac{1}{N_C^2}\right)\,. \tag{2.8}$$

It is worth noticing that we start our discussions from an S matrix theory point of view: The width has a non-perturbative definition and is related to the imaginary part of the pole position. This is very important since it enables us to investigate general properties of resonances without recurring to perturbative calculations of the width. As it will be seen later the resonance sum rules derived and investigated in this paper are obtained without making use of resonance chiral lagrangians of any kind. Only when we apply our relations in lagrangian models, the latter will be needed.

JHEP06(2007)030

2.1.2 Virtual pole in the $IJ = 20$ channel

Contrary to the $IJ = 00$ and $IJ = 11$ channels, the $IJ = 20$ S-matrix contains a virtual pole hidden on the second Riemann sheet at $s_v^{(20)}$, related to a S-matrix zero in the first Riemann sheet [8]. The pole position is estimated in the large-N_C limit from the χPT S-matrix, $S^{\chi PT}(s)_{(20)} = 1 + 2i\rho(s) T^{\chi PT}(s)_{(20)}$:

$$
\begin{aligned}
s_v^{(20)} &= 16\, m_\pi^2\, T^{\chi PT}(0)^2 \;+\; \mathcal{O}(m_\pi^{10}) \\
&= \frac{m_\pi^6}{16\pi^2 f^4} + \frac{m_\pi^8}{3\pi^2 f^6}(10L_2 + 2L_3 - 3L_5 + 6L_8) \;+\; \mathcal{O}(m_\pi^{10}) \,,
\end{aligned}
\tag{2.9}
$$

where $s_v^{(20)}$ is $\mathcal{O}(m_\pi^6)$ in the chiral expansion. The contribution of a virtual S-matrix pole can be parameterized as

$$
S^{sv}(s)_{(20)} \;=\; 1 \;+\; 2\, i\, \rho(s)\, T^{\rm sv}(s)_{(20)} \tag{2.10}
$$

with the T-matrix,

$$
T^{\rm sv}(s)_{(20)} \;=\; \frac{a_v^{(20)}}{1 \;-\; i\, \rho(s)\, a_v^{(20)}} \,. \tag{2.11}
$$

The scattering length $a_v^{(20)}$ is related to the virtual pole position through

$$
\begin{aligned}
a_v^{(20)} &= \sqrt{\frac{s_v^{(20)}}{4m_\pi^2 - s_v^{(20)}}} \;=\; 2\, T^{\chi PT}(0)_{(20)} \;+\; \mathcal{O}(m_\pi^6) \\
&= \frac{m_\pi^2}{8\pi f^2} + \frac{m_\pi^4}{3\pi f^4}(10L_2 + 2L_3 - 3L_5 + 6L_8) \;+\; \mathcal{O}(m_\pi^6) \,.
\end{aligned}
\tag{2.12}
$$

Hence, at leading order in $1/N_C$, the contribution to the $IJ = 20$ T-matrix from the virtual pole is

$$
T^{\rm sv}(s)_{(20)} \;=\; a_v^{(20)} \;+\; \mathcal{O}\left(\frac{1}{N_C^2}\right) \;=\; 2\, T^{\chi PT}(0)_{(20)} \;+\; \mathcal{O}\left(\frac{1}{N_C^2}, m_\pi^6\right) \,. \tag{2.13}
$$

2.2 Contribution from the t-channel resonance exchange

The contribution $S^{\rm cut}$ only contains cuts. It can be parameterized in the form [9],

$$
f(s) \;\equiv\; \frac{1}{2i\rho(s)} \;\ln S^{\rm cut}(s) \tag{2.14}
$$

and $f(s)$ satisfies the following once subtracted dispersion relation

$$
f(s) = f_L(s) + f_R(s) \equiv \frac{s}{\pi}\int_L \frac{{\rm Im}_L f(s')}{s'(s'-s)} ds' + \frac{s}{\pi}\int_{R'} \frac{{\rm Im}_R f(s')}{s'(s'-s)} ds' \,, \tag{2.15}
$$

where L denotes the left-hand cuts and R' denotes the inelastic cuts beyond the $\pi\pi$ elastic one. In the large N_C limit this reduces to a left-hand cut contribution from the t-channel resonance exchange if higher resonance multiplets are neglected in the s-channel .

Naively, one would expect the two-particle left-hand cuts to be subleading in $1/N_C$. However, $\mathrm{Im}_L f(s)$ contains a kinematical singularity at $s = 0$. As the dispersive left-hand $\pi\pi$ cut runs in the range $(-\infty, 0]$, one gets the contribution [5]

$$f(s)_{L,\pi\pi} \;=\; -\,|T(0)| \,+\, \mathcal{O}(1/N_C^2)\,, \tag{2.16}$$

with $T(0)$ the value of the physical T-matrix at $s = 0$. The discontinuity of $f(s)$ for the left-hand cut due to the t-channel resonance exchange obeys the relation

$$\mathrm{Im}_L f(s) = -\frac{1}{2\rho(s)}\ln|S^{\rm cut}(s)| = -\frac{1}{2\rho(s)}\ln|S(s)| = \mathrm{Im}_L T \,+\, \mathcal{O}\left(\frac{1}{N_C^2}\right)\,, \tag{2.17}$$

where $\ln|S(s)| = \frac{1}{2}\ln\left[1 - 4\rho(s)\mathrm{Im}_L T(s) + 4\rho^2(s)|T(s)|^2\right]$ has been expanded using $T(s) = \mathcal{O}\left(\frac{1}{N_C}\right)$. Since the cut due to crossed channel resonance exchanges does not contain the singular point $s = 0$, the expansion of the logarithm in $1/N_C$ can be safely performed. By means of eqs. (2.15)–(2.17), one finds the left-hand cut contribution to be given by

$$f_L(s) \;=\; -\,|T(0)| + \sum_R T^{\rm tR}(s) \;+\; \mathcal{O}\left(\frac{1}{N_C^2}\right)\,, \tag{2.18}$$

with the t-channel resonance exchange contribution,

$$T^{\rm tR}(s) \;=\; \frac{s}{\pi}\int_{-\infty}^{-M_{\rm R}^2+4m_\pi^2}\frac{\mathrm{Im}T^{\rm tR}(s')}{s'(s'-s)}ds'\,. \tag{2.19}$$

According to the convention provided by ref. [6], the left-hand cut, or the background contribution to the scattering phase shift is,

$$\delta_{BG} = \rho(s) f_L(s)\,. \tag{2.20}$$

From eq. (2.16), at large-N_C, there is always a negative contribution $-|T(0)|$ to the scattering lengths. On the other hand, the contribution from the crossed channel large-N_C resonances varies in different channels. This will be further discussed in section 3.1.

Crossing symmetry relates the right to the left-hand cut through the expression [10],

$$\begin{aligned}\mathrm{Im}_{\rm L} T_J^I(s) \;=\; & \frac{1+(-1)^{I+J}}{s-4m_\pi^2}\sum_{J'}\sum_{I'}(2J'+1)C_{II'}^{st} \\ & \times\int_{4m_\pi^2}^{4m_\pi^2-s} dt\; P_J\left(1+\frac{2t}{s-4m_\pi^2}\right)P_{J'}\left(1+\frac{2s}{t-4m_\pi^2}\right)\mathrm{Im}_{\rm R} T_{J'}^{I'}(t)\,,\end{aligned} \tag{2.21}$$

with $P_n(x)$ the Legendre polynomials. In general, this representation is only valid for the range $-32m_\pi^2 < s < 0$ if the Mandelstam representation is assumed [10]. Nevertheless, in the large-N_C limit, eq. (2.21) actually work for any energy since the double spectral function vanishes at this order of the $1/N_C$ expansion. The crossing matrix is given by [10]

$$C_{II'}^{(st)} \;=\; \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}\,. \tag{2.22}$$

Substituting the narrow-width right-hand cut expression from eq. (2.7), one gets the contribution from the t-channel exchange of a resonance R with $I'J'$ quantum numbers:

$$\begin{aligned}\mathrm{Im}T^{\mathrm{tR}}(s)^I_J = \;& \theta(-s-M_{\mathrm{R}}^2+4m_\pi^2)\;\times\;\frac{1+(-1)^{I+J}}{s-4m_\pi^2}(2J'+1)C^{st}_{II'} \\ & \times\, P_J\left(1+\frac{2M_{\mathrm{R}}^2}{s-4m_\pi^2}\right)P_{J'}\left(1+\frac{2s}{M_{\mathrm{R}}^2-4m_\pi^2}\right)\frac{\pi\,M_{\mathrm{R}}\,\Gamma_{\mathrm{R}}}{\rho(M_{\mathrm{R}}^2)}\,.\end{aligned} \tag{2.23}$$

In our analysis, only vector and scalar resonances are considered. Their contributions to the different channels are obtained through eq. (2.19):

(i) $IJ=11$ channel

$$\begin{aligned}T^{\mathrm{tS}}(s) =\;& \frac{2M_{\mathrm{S}}\Gamma_{\mathrm{S}}}{3\rho(M_{\mathrm{S}}^2)}\left[\frac{-s}{2m_\pi^2(s-4m_\pi^2)}+\frac{2m_\pi^2-M_{\mathrm{S}}^2}{8m_\pi^4}\ln\frac{M_{\mathrm{S}}^2-4m_\pi^2}{M_{\mathrm{S}}^2}\right. \\ & \left.+\frac{s+2M_{\mathrm{S}}^2-4m_\pi^2}{(s-4m_\pi^2)^2}\ln\frac{s+M_{\mathrm{S}}^2-4m_\pi^2}{M_{\mathrm{S}}^2}\right],\end{aligned} \tag{2.24}$$

$$\begin{aligned}T^{\mathrm{tV}}(s) =\;& \frac{3M_{\mathrm{V}}\Gamma_{\mathrm{V}}}{\rho(M_{\mathrm{V}}^2)}\left[\frac{-s(M_{\mathrm{V}}^2+4m_\pi^2)}{2m_\pi^2(s-4m_\pi^2)(M_{\mathrm{V}}^2-4m_\pi^2)}\right. \\ & +\frac{8m_\pi^4-6m_\pi^2M_{\mathrm{V}}^2+M_{\mathrm{V}}^4}{8m_\pi^4(4m_\pi^2-M_{\mathrm{V}}^2)}\ln\frac{M_{\mathrm{V}}^2-4m_\pi^2}{M_{\mathrm{V}}^2} \\ & +\frac{16m_\pi^4-12m_\pi^2s-12m_\pi^2M_{\mathrm{V}}^2+5M_{\mathrm{V}}^2s+2M_{\mathrm{V}}^4+2s^2}{(s-4m_\pi^2)^2(M_{\mathrm{V}}^2-4m_\pi^2)} \\ & \left.\times\ln\frac{s+M_{\mathrm{V}}^2-4m_\pi^2}{M_{\mathrm{V}}^2}\right],\end{aligned} \tag{2.25}$$

(ii) $IJ=00$ channel

$$T^{\mathrm{tS}}(s) = \frac{2M_{\mathrm{S}}\Gamma_{\mathrm{S}}}{3\rho(M_{\mathrm{S}}^2)}\left[\frac{1}{4m_\pi^2}\ln\frac{M_{\mathrm{S}}^2-4m_\pi^2}{M_{\mathrm{S}}^2}+\frac{1}{s-4m_\pi^2}\ln\frac{s+M_{\mathrm{S}}^2-4m_\pi^2}{M_{\mathrm{S}}^2}\right], \tag{2.26}$$

$$\begin{aligned}T^{\mathrm{tV}}(s) =\;& \frac{6M_{\mathrm{V}}\Gamma_{\mathrm{V}}}{\rho(M_{\mathrm{V}}^2)}\left[\frac{1}{4m_\pi^2}\ln\frac{M_{\mathrm{V}}^2-4m_\pi^2}{M_{\mathrm{V}}^2}+\frac{2s+M_{\mathrm{V}}^2-4m_\pi^2}{(s-4m_\pi^2)(M_{\mathrm{V}}^2-4m_\pi^2)}\right. \\ & \left.\times\ln\frac{s+M_{\mathrm{V}}^2-4m_\pi^2}{M_{\mathrm{V}}^2}\right],\end{aligned} \tag{2.27}$$

(iii) $IJ=20$ channel

$$T^{\mathrm{tS}}(s) = \frac{2M_{\mathrm{S}}\Gamma_{\mathrm{S}}}{3\rho(M_{\mathrm{S}}^2)}\left[\frac{1}{4m_\pi^2}\ln\frac{M_{\mathrm{S}}^2-4m_\pi^2}{M_{\mathrm{S}}^2}+\frac{1}{s-4m_\pi^2}\ln\frac{s+M_{\mathrm{S}}^2-4m_\pi^2}{M_{\mathrm{S}}^2}\right], \tag{2.28}$$

$$\begin{aligned}T^{\mathrm{tV}}(s) =\;& \frac{-3M_{\mathrm{V}}\Gamma_{\mathrm{V}}}{\rho(M_{\mathrm{V}}^2)}\left[\frac{1}{4m_\pi^2}\ln\frac{M_{\mathrm{V}}^2-4m_\pi^2}{M_{\mathrm{V}}^2}+\frac{2s+M_{\mathrm{V}}^2-4m_\pi^2}{(s-4m_\pi^2)(M_{\mathrm{V}}^2-4m_\pi^2)}\right. \\ & \left.\times\ln\frac{s+M_{\mathrm{V}}^2-4m_\pi^2}{M_{\mathrm{V}}^2}\right],\end{aligned} \tag{2.29}$$

where T^{tS} and T^{tV} denote the contributions from scalar and vector resonances, respectively.

2.3 Summation of right- and left-hand cuts

Putting all the different contributions at leading order in $1/N_C$ together one gets

$$S(s) \;=\; S^{\rm cut}(s)\cdot\prod_{\rm R} S^{\rm R}(s) \;=\; 1\,+\,2\,i\,\rho(s)\,T(s)_{N_C\to\infty} \;+\; \mathcal{O}\left(\frac{1}{N_C^2}\right)\,, \tag{2.30}$$

with the large-N_C T-matrix given by

$$T(s)_{N_C\to\infty} \;=\; \sum_{\rm R} T^{\rm sR}(s)\,+\,T^{\rm sv}(s)\,-\,|T(0)|\,+\,\sum_{\rm R} T^{\rm tR}(s)\,. \tag{2.31}$$

This expression can be simplified taking into account that, in the channels $IJ=00$ and $IJ=11$, there is no virtual pole ($T^{\rm sv}(s)=0$) and χPT tells us that $|T(0)|=-T(0)$. In the $IJ=20$ case, χPT dictates $|T(0)|=T(0)$ and the virtual pole contribution $T^{\rm sv}(s)=2\,T(0)$. Thus, eq. (2.31) can be rewritten in the way

$$T(s)_{N_C\to\infty} \;=\; T(0)\,+\,\sum_{R} T^{\rm tR}(s)\,+\,\sum_{R} T^{\rm sR}(s)\,. \tag{2.32}$$

An alternative way to reach this relation is through the T-matrix dispersive relation

$$T(s) \;=\; T(0)\,+\,\frac{s}{\pi}\int_{-\infty}^{0}\frac{ds'\,{\rm Im}T(s')}{s'(s'-s)}\;+\;\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'\,{\rm Im}T(s')}{s'(s'-s)}\,. \tag{2.33}$$

The above derivation demonstrates that the dispersive parametrization in eq. (2.1) [5] is equivalent to a T-matrix partial wave dispersion relation under narrow width approximation. The PKU parametrization form is, in this sense, simply a combination of partial wave dispersion relation and single channel unitarity.

3. Low-energy matching

3.1 Low-energy expansion of the s- and t-channel resonance contributions

We now intend to perform a matching of our dispersive expression in eq. (2.32) to low-energy QCD, provided by Chiral Perturbation Theory (χPT) [3]. Hence, we perform a threshold expansion in the form

$$A(s) = \sum_{n=0}^{\infty} a_{2n}\left(\frac{s-4m_\pi^2}{m_\pi^2}\right)^n\,. \tag{3.1}$$

The constants a_n are functions of m_π^2 and can be also chiral expanded in the form

$$a_{2n} \;=\; \sum_{k=0}^{\infty} a_{2n,2k}\,(m_\pi^2)^k\,. \tag{3.2}$$

To match χPT up to a given order $\mathcal{O}\left(p^{2\ell}\right)$ means to match the corresponding coefficients $a_{2n,2k}$ for $n=0\ldots\ell$, $k\le\ell-n$.

Taking the result from eq. (2.32) to low energies and matching χPT leads to the relation

$$t_0^{\chi PT} - T^{\chi PT}(0) + t_2^{\chi PT}\left(\frac{s-4m_\pi^2}{m_\pi^2}\right) + t_4^{\chi PT}\left(\frac{s-4m_\pi^2}{m_\pi^2}\right)^2 + \cdots \tag{3.3}$$
$$= [t_0^s + t_0^t] + [t_2^s + t_2^t]\left(\frac{s-4m_\pi^2}{m_\pi^2}\right) + [t_4^s + t_4^t]\left(\frac{s-4m_\pi^2}{m_\pi^2}\right)^2 + \cdots$$

The scattering amplitude $T(s)$ on the left-hand side of eq. (2.33) and $T(0)$ have been substituted by their value in χPT. The matching is performed in this work up to $\mathcal{O}(p^4)$. The expansion of the right-hand cut contribution $\sum_R T^{\rm sR}(s) = \sum_n t_{2n}^s \left(\frac{s-4m_\pi^2}{m_\pi^2}\right)^n$ is provided by the coefficients

$$t_0^s = \sum_R \frac{1}{\rho(M_{\rm R}^2)^3}\,\frac{\Gamma_{\rm R}}{M_{\rm R}}\,\frac{4m_\pi^2}{M_{\rm R}^2},$$
$$t_{2n}^s = \sum_R \frac{1}{\rho(M_{\rm R}^2)^{2n+3}}\,\frac{\Gamma_{\rm R}}{M_{\rm R}}\left(\frac{m_\pi^2}{M_{\rm R}^2}\right)^n, \qquad \text{for } n\geq 1\,. \tag{3.4}$$

The subscript R denotes the resonances R with the appropriate IJ quantum numbers of the channel. Only one multiplet of scalars and vector mesons is considered in the present study.

Up to $\mathcal{O}(p^4)$, the expansion of the t-channel resonance exchange $\sum_R T^{\rm tR}(s) = \sum_n t_{2n}^t \left(\frac{s-4m_\pi^2}{m_\pi^2}\right)^n$ yields

(i) $IJ=11$ channel

$$t_0^t = \left(\frac{4\Gamma_{\rm S}}{9M_{\rm S}^3} + \frac{2\Gamma_{\rm V}}{M_{\rm V}^3}\right)m_\pi^2 + \left(\frac{8\Gamma_{\rm S}}{3M_{\rm S}^5} + \frac{12\Gamma_{\rm V}}{M_{\rm V}^5}\right)m_\pi^4,$$
$$t_2^t = \left(\frac{\Gamma_{\rm S}}{9M_{\rm S}^3} + \frac{\Gamma_{\rm V}}{2M_{\rm V}^3}\right)m_\pi^2 + \left(\frac{2\Gamma_{\rm S}}{9M_{\rm S}^5} + \frac{5\Gamma_{\rm V}}{M_{\rm V}^5}\right)m_\pi^4,$$
$$t_4^t = \left(\frac{-\Gamma_{\rm S}}{9M_{\rm S}^5} + \frac{\Gamma_{\rm V}}{2M_{\rm V}^5}\right)m_\pi^4. \tag{3.5}$$

(ii) $IJ=00$ channel

$$t_0^t = \left(\frac{-4\Gamma_{\rm S}}{3M_{\rm S}^3} + \frac{36\Gamma_{\rm V}}{M_{\rm V}^3}\right)m_\pi^2 + \left(\frac{-56\Gamma_{\rm S}}{9M_{\rm S}^5} + \frac{232\Gamma_{\rm V}}{M_{\rm V}^5}\right)m_\pi^4,$$
$$t_2^t = \left(\frac{-\Gamma_{\rm S}}{3M_{\rm S}^3} + \frac{9\Gamma_{\rm V}}{M_{\rm V}^3}\right)m_\pi^2 + \left(\frac{-2\Gamma_{\rm S}}{3M_{\rm S}^5} + \frac{42\Gamma_{\rm V}}{M_{\rm V}^5}\right)m_\pi^4,$$
$$t_4^t = \left(\frac{2\Gamma_{\rm S}}{9M_{\rm S}^5} - \frac{4\Gamma_{\rm V}}{M_{\rm V}^5}\right)m_\pi^4. \tag{3.6}$$

(iii) $IJ=20$ channel

$$t_0^t = -\left(\frac{4\Gamma_{\rm S}}{3M_{\rm S}^3} + \frac{18\Gamma_{\rm V}}{M_{\rm V}^3}\right)m_\pi^2 - \left(\frac{56\Gamma_{\rm S}}{9M_{\rm S}^5} + \frac{116\Gamma_{\rm V}}{M_{\rm V}^5}\right)m_\pi^4,$$
$$t_2^t = -\left(\frac{\Gamma_{\rm S}}{3M_{\rm S}^3} + \frac{9\Gamma_{\rm V}}{2M_{\rm V}^3}\right)m_\pi^2 - \left(\frac{2\Gamma_{\rm S}}{3M_{\rm S}^5} + \frac{21\Gamma_{\rm V}}{M_{\rm V}^5}\right)m_\pi^4,$$
$$t_4^t = \left(\frac{2\Gamma_{\rm S}}{9M_{\rm S}^5} + \frac{2\Gamma_{\rm V}}{M_{\rm V}^5}\right)m_\pi^4\,. \tag{3.7}$$

The quantities t_0^t and t_0^s in each channel actually gives, respectively, the crossed-channel and the s-channel resonance contribution to the scattering length parameter.

3.2 Chiral perturbation theory scattering amplitude

In the large-N_C limit, the χPT scattering amplitude is given up to $\mathcal{O}(p^4)$ by the coefficients [4]:

(i) $IJ = 11$ channel

$$\begin{aligned} t_0^{\chi PT} &= 0, \\ t_2^{\chi PT} &= \frac{m_\pi^2}{96\pi f^2} - \frac{m_\pi^4}{6\pi f^4} L_3 \,, \\ t_4^{\chi PT} &= \frac{-m_\pi^4}{24\pi f^4} L_3 \,, \\ T(0)^{\chi PT} &= \frac{-m_\pi^2}{24\pi f^2} \,. \end{aligned} \tag{3.8}$$

(ii) $IJ = 00$ channel

$$\begin{aligned} t_0^{\chi PT} &= \frac{7m_\pi^2}{32\pi f^2} + \frac{m_\pi^4}{2\pi f^4}\left(15L_2 + 5L_3 - \frac{5}{2}L_5 + 5L_8\right), \\ t_2^{\chi PT} &= \frac{m_\pi^2}{16\pi f^2} + \frac{m_\pi^4}{\pi f^4}(5L_2 + 2L_3)\,, \\ t_4^{\chi PT} &= \frac{m_\pi^4}{24\pi f^4}(25L_2 + 11L_3)\,, \\ T(0)^{\chi PT} &= \frac{-m_\pi^2}{32\pi f^2} + \frac{m_\pi^4}{6\pi f^4}\left(25L_2 + 11L_3 - \frac{15}{2}L_5 + 15L_8\right). \end{aligned} \tag{3.9}$$

(iii) for IJ=20 channel one has:

$$\begin{aligned} t_0^{\chi PT} &= \frac{-m_\pi^2}{16\pi f^2} + \frac{m_\pi^4}{\pi f^4}\left(3L_2 + L_3 - \frac{1}{2}L_5 + L_8\right), \\ t_2^{\chi PT} &= \frac{-m_\pi^2}{32\pi f^2} + \frac{m_\pi^4}{2\pi f^4}(4L_2 + L_3)\,, \\ t_4^{\chi PT} &= \frac{m_\pi^4}{12\pi f^4}(5L_2 + L_3)\,, \\ T(0)^{\chi PT} &= \frac{m_\pi^2}{16\pi f^2} + \frac{m_\pi^4}{3\pi f^4}\left(5L_2 + L_3 - \frac{3}{2}L_5 + 3L_8\right). \end{aligned} \tag{3.10}$$

where the $t_n^{\chi PT}$ are given by the threshold expansion of the chiral amplitude $T(s)^{\chi PT} = \sum_{n=0}^{\infty} t_{2n}^{\chi PT}\left(\frac{s-4m_\pi^2}{m_\pi^2}\right)^n$ and $T(0)^{\chi PT}$ denotes the value of the χPT scattering amplitude at $s = 0$. The constant f is the chiral limit of the pion decay constant, $f \approx 88\,\mathrm{MeV}$ [3]. In order to get the expressions in eqs. (3.8)–(3.10), the one-loop contributions have been dropped and we have made use of the large-N_C relations $L_4 = L_6 = 0$ and $L_1 = L_2/2$ [4].

3.3 Matching dispersive and χPT expressions

Having obtained the resonance expansions as well as the chiral expansions at threshold, matching conditions can be set up between the two kind of amplitudes. For simplicity we in the following only introduce minimal set of resonances, i.e., only σ and ρ. We point out that in case of need it is straightforward to add higher resonances in the present scheme.

The matching in eq. (3.3), considered order by order in the threshold expansion, leads to a series of relations. Only the terms up to $\mathcal{O}(p^4)$ in the chiral expansion are retained in this work:

(i) $IJ = 11$ channel

$$\frac{1}{24\pi f^2} = \frac{4\Gamma_{\rm S}}{9M_{\rm S}^3} + \frac{6\Gamma_{\rm V}}{M_{\rm V}^3} + \left(\frac{8\Gamma_{\rm S}}{3M_{\rm S}^5} + \frac{36\Gamma_{\rm V}}{M_{\rm V}^5}\right) m_\pi^2\,, \tag{3.11}$$

$$\frac{1}{96\pi f^2} - \frac{m_\pi^2}{6\pi f^4}L_3 = \frac{\Gamma_{\rm S}}{9M_{\rm S}^3} + \frac{3\Gamma_{\rm V}}{2M_{\rm V}^3} + \left(\frac{2\Gamma_{\rm S}}{9M_{\rm S}^5} + \frac{15\Gamma_{\rm V}}{M_{\rm V}^5}\right) m_\pi^2, \tag{3.12}$$

$$-\frac{L_3}{24\pi f^4} = -\frac{\Gamma_{\rm S}}{9M_{\rm S}^5} + \frac{3\Gamma_{\rm V}}{2M_{\rm V}^5}, \tag{3.13}$$

(ii) $IJ = 00$ channel

$$\frac{1}{4\pi f^2} + \frac{m_\pi^2}{3\pi f^4}(10L_2 + 2L_3) = \frac{8\Gamma_{\rm S}}{3M_{\rm S}^3} + \frac{36\Gamma_{\rm V}}{M_{\rm V}^3} + \left(\frac{160\Gamma_{\rm S}}{9M_{\rm S}^5} + \frac{232\Gamma_{\rm V}}{M_{\rm V}^5}\right) m_\pi^2\,, \tag{3.14}$$

$$\frac{1}{16\pi f^2} + \frac{m_\pi^2}{\pi f^4}(5L_2 + 2L_3) = \frac{2\Gamma_{\rm S}}{3M_{\rm S}^3} + \frac{9\Gamma_{\rm V}}{M_{\rm V}^3} + \left(\frac{28\Gamma_{\rm S}}{3M_{\rm S}^5} + \frac{42\Gamma_{\rm V}}{M_{\rm V}^5}\right) m_\pi^2\,, \tag{3.15}$$

$$\frac{1}{24\pi f^4}(25L_2 + 11L_3) = \frac{11\Gamma_{\rm S}}{9M_{\rm S}^5} - \frac{4\Gamma_{\rm V}}{M_{\rm V}^5}, \tag{3.16}$$

(iii) $IJ = 20$ channel

$$-\frac{1}{8\pi f^2} + \frac{m_\pi^2}{3\pi f^4}(4L_2 + 2L_3) = -\frac{4\Gamma_{\rm S}}{3M_{\rm S}^3} - \frac{18\Gamma_{\rm V}}{M_{\rm V}^3} - \left(\frac{56\Gamma_{\rm S}}{9M_{\rm S}^5} + \frac{116\Gamma_{\rm V}}{M_{\rm V}^5}\right) m_\pi^2, \tag{3.17}$$

$$-\frac{1}{32\pi f^2} + \frac{m_\pi^2}{2\pi f^4}(4L_2 + L_3) = -\frac{\Gamma_{\rm S}}{3M_{\rm S}^3} - \frac{9\Gamma_{\rm V}}{2M_{\rm V}^3} - \left(\frac{2\Gamma_{\rm S}}{3M_{\rm S}^5} + \frac{21\Gamma_{\rm V}}{M_{\rm V}^5}\right) m_\pi^2\,, \tag{3.18}$$

$$\frac{(5L_2 + L_3)}{12\pi f^4} = \frac{2\Gamma_{\rm S}}{9M_{\rm S}^5} + \frac{2\Gamma_{\rm V}}{M_{\rm V}^5}\,. \tag{3.19}$$

A global factor m_π^2 has been simplified in eqs. (3.11), (3.12), (3.14), (3.15), (3.17) and (3.18), and eqs. (3.13), (3.16) and (3.19) have been divided by a factor m_π^4. Notice that the matching equations do not depend explicitly on the low-energy couplings L_5 and L_8. The contribution from the L_5 π^4 operator to the scattering amplitude is canceled out up to a constant term by the L_5 part of the pion wave function renormalization Z_π of the external legs. The L_8 operator does not contain derivatives and it just adds another energy independent term to the $\pi\pi$-amplitude. Since the constant contributions vanish when considering the difference $T(4m_\pi^2) - T(0)$ (with $T(4m_\pi^2) = t_0$ in our notation), L_5 and L_8 do no longer appear explicitly in the matching equations.

JHEP06(2007)030

The first thing to notice is that the identities related to the matching $t_0^{\chi PT} - T^{\chi PT}(0) = t_0^s + t_0^t$ (eqs. (3.11), (3.14) and (3.17)) are linear combinations of the other two matching relations for $t_2^{\chi PT}$ and $t_4^{\chi PT}$. This is due to the fact that $T^{\chi PT}(s) - T^{\chi PT}(0)$ vanishes at zero by construction. Hence, its threshold expansion carries the implicit relation $t_0^{\chi PT} - T^{\chi PT}(0) = 4t_2^{\chi PT} - 16t_4^{\chi PT}$ in our notation.

The physical widths and masses, $\Gamma_{\rm R}$ and $M_{\rm R}$, carry an implicit dependence on m_π^2, which can be expressed in the form

$$\frac{\Gamma_{\rm R}}{M_{\rm R}^3} = \frac{\Gamma_{\rm R}^{(0)}}{M_{\rm R}^{(0)\,3}} \left[1 + \alpha_R \frac{m_\pi^2}{M_{\rm R}^{(0)\,2}} + \mathcal{O}\left(m_\pi^4\right) \right] . \tag{3.20}$$

The constants $M_{\rm R}^{(0)}$ and $\Gamma_{\rm R}^{(0)}$ are respectively the mass and width of the resonance R in the chiral limit and α_R parameterizes the deviation from the chiral limit.

The matching to χPT at $\mathcal{O}(p^2)$ is given by the $\mathcal{O}(m_\pi^0)$ terms in eqs. (3.12), (3.15) and (3.18). The three different channels produce the same equation,

$$\frac{1}{16\pi f^2} = \frac{9\Gamma_V^{(0)}}{M_{\rm V}^{(0)\,3}} + \frac{2\Gamma_S^{(0)}}{3M_{\rm S}^{(0)\,3}}, \tag{3.21}$$

which is nothing but a extension to the well known KSRF relation [11].

One old way to express the KSRF relation is the following,

$$g_{\rho\pi\pi}^2 = \frac{M_\rho^2}{2f_\pi^2}, \tag{3.22}$$

where $g_{\rho\pi\pi}$ characterizes the $\rho - \pi\pi$ coupling. For a massive Yang-Mills model, the chiral limit of the ρ width is given by

$$\Gamma_\rho = \frac{g_{\rho\pi\pi}^2}{48\pi} M_\rho . \tag{3.23}$$

Combining eqs. (3.22) and (3.23) leads to

$$\frac{1}{16\pi f^2} = \frac{6\Gamma_V^{(0)}}{M_{\rm V}^{(0)\,3}} . \tag{3.24}$$

The difference between eqs. (3.21) and (3.24) on the *r.h.s.* is clearly understood when we examine the matching in the IJ=11 channel: it comes from the crossed channel vector and scalar meson exchanges, which is absent in eq. (3.24). Furthermore, it is remarkable to notice that, all the three channels lead to the same generalized KSRF relation. The modification of the KSRF relation due to the crossed channel resonance exchange was first noticed in ref. [12].[1] Our work stressed that the correct expression of the so-called KSRF relation can be obtained in a systematic way without relying on any particular lagrangian formalism: once subtracted partial wave dispersion relations combined with chiral symmetry and large N_C expansion (or narrow width approximation) generates our modified

JHEP06(2007)030

	$T(0)$	$t_0^{\rm tR}$	$t_0^{\rm sR}$	$t_0^{\chi PT}$
$IJ=11$	$-\frac{m_\pi^2}{24\pi f^2}$	$\frac{4\Gamma_S}{9M_S^3}+\frac{2\Gamma_V}{M_V^3}$	$\frac{4\Gamma_V}{M_V^3}$	0
$IJ=00$	$-\frac{m_\pi^2}{32\pi f^2}$	$-\frac{4\Gamma_S}{3M_S^3}+\frac{36\Gamma_V}{M_V^3}$	$\frac{4\Gamma_S}{M_S^3}$	$\frac{7m_\pi^2}{32\pi f^2}$
$IJ=20$	$\frac{m_\pi^2}{16\pi f^2}$	$-\frac{4\Gamma_S}{3M_S^3}-\frac{18\Gamma_V}{M_V^3}$	0	$-\frac{m_\pi^2}{16\pi f^2}$

Table 1: Summary of the different contributions $T(0)$, $t_0^{\rm tR}$, $t_0^{\rm SR}$ to the scattering lengths at leading order in the m_π^2 expansion. The generalized KSRF-relation derives from the matching of the sum of the first three columns to the χPT prediction, $t_0^{\chi PT}$. In the last line, $T(0)$ contains the sum of $-|T(0)|$ and the $IJ=20$ virtual pole contribution.

KSRF relation. The matching at both high and low energies are crucial for establishing this constraint. The different contributions to the KSRF relation are summarized in table 1.

The matching to χPT at $\mathcal{O}(p^4)$, gives another six identities. The $\mathcal{O}\left((s-4m_\pi^2)^2\right)$ terms from the $IJ=11,00,20$ channels (eqs. (3.13), (3.16) and (3.19)) provide the constraints

$$L_2 \;=\; 12\pi f^4 \frac{\Gamma_{\rm V}^{(0)}}{M_{\rm V}^{(0)\,5}}\,, \tag{3.25}$$

$$L_3 \;=\; 4\pi f^4 \left(\frac{2\Gamma_{\rm S}^{(0)}}{3M_{\rm S}^{(0)\,5}}-\frac{9\Gamma_{\rm V}^{(0)}}{M_{\rm V}^{(0)\,5}}\right)\,. \tag{3.26}$$

The eqs. (3.25), (3.26) provide a large N_C prediction for the LECs L_2 and L_3. The two expressions obey the positivity constraints: $L_2>0$ and $3L_2+L_3>0$ as revealed in ref. [14].

The remaining $\mathcal{O}(p^4)$ relations are provided by the $\mathcal{O}(m_\pi^2)$ terms in the $\mathcal{O}(s-4m_\pi^2)$ equations (eqs. (3.12), (3.15) and (3.18)), and produce

$$0=\frac{2}{3}\,\frac{\Gamma_{\rm S}^{(0)}}{M_{\rm S}^{(0)\,5}}\,[\alpha_S+6]\,+\,\frac{9\,\Gamma_{\rm V}^{(0)}}{M_{\rm V}^{(0)\,5}}\,[\alpha_V+6]\,. \tag{3.27}$$

The novel relation, eq. (3.27) casts an interesting relation between resonance parameters. The eqs. (3.25), (3.26), (3.27) and the extended KSRF relation, eq. (3.21) are generated simultaneously, in a systematic way, by a matching to χPT amplitude at different chiral orders. The following section is devoted to a better understanding to the new relation, eq. (3.27).

4. On the consistency of lagrangian models

In this section, we inspect several phenomenological lagrangians that have been proposed in order to describe the resonance interactions. Firstly, we will consider the toy model with a linear sigma meson representation and the chiral gauged model [15], which only

[1] Instead of eq. (3.22), the relation given in ref. [12] is, $g_{\rho\pi\pi}^2=\frac{M_\rho^2}{3f_\pi^2}$. In ref. [13], Hikasa and Igi included scalar exchange and were able to obtain a relation similar to eq. (3.21) in all three channels, assisted by N/D method.

introduces vector mesons. These examples illustrate very nicely the expected properties that a meson theory must fulfill. A similar analysis can be also carried within the hidden local symmetry model [16]. We end the section with an extensive analysis within resonance chiral theory [17, 18].

4.1 Linear sigma model

The linear sigma model (LσM) with massive pions is given by the lagrangian

$$\mathcal{L}_{L\sigma M} \,=\, \frac{1}{2}\left[(\partial\pi)^2+(\partial\sigma)^2\right] \,+\, \frac{1}{2}\mu^2\left[\pi^2+\sigma^2\right] \,-\, \frac{1}{4}\lambda\left[\pi^2+\sigma^2\right]^2 \,+\, f\, m_\pi^2\,\sigma\,, \tag{4.1}$$

with $f=\sqrt{\frac{\mu^2}{\lambda}}$. No vectors are considered in this model.

After shifting the σ field due to its vacuum expectation value $\langle\sigma\rangle=\sqrt{\frac{\mu^2}{\lambda}}\left(1+\frac{m_\pi^2}{\mu^2}+\ldots\right)$, one gets the tree-level mass term

$$M_\sigma^2 \,=\, M_\sigma^{(0)\,2}\left[1+\frac{3\,m_\pi^2}{M_\sigma^{(0)\,2}}+\ldots\right]\,, \tag{4.2}$$

with $M_\sigma^{(0)\,2}=2\,\mu^2$. The large-$N_C$ width is given by the $\sigma-\pi\pi$ vertex:

$$\Gamma_\sigma \,=\, \Gamma_\sigma^{(0)}\left[1-\frac{3\,m_\pi^2}{2\,M_\sigma^{(0)\,2}}+\ldots\right]\,, \tag{4.3}$$

with $\Gamma_\sigma^{(0)}=\frac{3\lambda}{16\pi}M_\sigma^{(0)}$. Putting both expressions together in the combination Γ_σ/M_σ^3 one gets

$$\alpha_S \,=\, \left[\frac{M_\sigma^5}{\Gamma_\sigma}\frac{d}{dm_\pi^2}\left(\frac{\Gamma_\sigma}{M_\sigma^3}\right)\right]_{m_\pi^2=0} \,=\, -6\,. \tag{4.4}$$

Since there are no vectors in the theory, eq. (3.27) is exactly fulfilled. Likewise, the LσM produce the value

$$\frac{\Gamma_\sigma^{(0)}}{M_\sigma^{(0)\,3}} \,=\, \frac{3\,\lambda}{32\pi\mu^2} \,=\, \frac{3}{32\pi f^2}\,. \tag{4.5}$$

Since there are no vectors in the theory, this result fulfills the modified-KSRF relation in eq. (3.21) for any value of the couplings μ and λ.

This can be better understood through the explicit diagramatic calculation. The analysis of the $\pi\pi$-scattering amplitude shows that the structure of the LσM lagrangian ensures a good high energy behaviour, independently of the value of the resonance parameters. Since the model obeys the proper high and low energy limits by construction, no resonance constraint can be extracted, just the usual low-energy coupling determinations for L_2 and L_3.

This exercise shows how, in order to fulfill the former constraints, a theory must have a right asymptotic behavior at high and low energies. In this case, chiral invariance ensures the right low energy properties and the LσM renormalizability ensures the proper high energy asymptotic behavior. However, the next example shows that renormalizability is not actually the necessary condition for the fulfillment of our large-N_C sum-rules.

4.2 The gauged chiral model

In this model, vector and axial-vector resonances are included as gauge bosons in the SU(2) χPT lagrangian [15]:

$$\mathcal{L}_{G\chi M} = \frac{f_0^2}{4}\langle D_\mu U D^\mu U^+\rangle + \frac{m_\pi^2 f^2}{4}\langle U + U^\dagger\rangle - \frac{1}{4}\langle L_{\mu\nu}L^{\mu\nu} + R_{\mu\nu}R^{\mu\nu}\rangle$$
$$+M_0^2\langle L_\mu L^\mu + R_\mu R^\mu\rangle + B\langle L_\mu U R^\mu U^+\rangle, \tag{4.6}$$

with $\langle\ldots\rangle$ short for trace in flavor space. The chiral tensors are defined as

$$\begin{aligned} U &= \exp\left(i\frac{\tau^a\pi^a}{f}\right), \\ D_\mu U &= \partial_\mu U - igL_\mu U + igUR_\mu, \\ L_\mu &= \frac{\tau^a}{2}(V_\mu^a + A_\mu^a), \\ R_\mu &= \frac{\tau^a}{2}(V_\mu^a - A_\mu^a), \\ L_{\mu\nu} &= \partial_\mu L_\nu - \partial_\nu L_\mu - ig[L_\mu, L_\nu], \\ R_{\mu\nu} &= \partial_\mu R_\nu - \partial_\nu R_\mu - ig[R_\mu, R_\nu], \end{aligned} \tag{4.7}$$

where V_μ^a and A_μ^a are the SU(2) vector and axial-vector triplets, respectively, ρ and a_1, and τ^a are the Pauli matrices. The last term, with coefficient B, is not essential and allows the model to be reasonably compatible with phenomenology [15]. It is dropped off in our analysis, following the derivation in the original paper.

The calculation the tree level $\rho \to \pi\pi$ decay width and the low-energy $\pi\pi$-scattering amplitude casts,

$$\Gamma_{\rho\to\pi\pi} = \frac{g_\rho^2\, M_\rho}{48\pi}\,\rho(M_\rho^2)^3\,, \tag{4.8}$$

$$L_2 = \frac{g_\rho^2 f^4}{4\,M_\rho^4}\,, \tag{4.9}$$

$$L_3 = \frac{3\,g_\rho^2 f^4}{4\,M_\rho^4}\,, \tag{4.10}$$

where the parameters g_ρ, f, M_ρ are related to the original couplings in the lagrangian through[2]

$$g_\rho^2 = g^2\left(1 - \frac{g^2 f^2}{4M_0^2}\right)^2, \tag{4.11}$$

$$f^2 = f_0^2\left(1 + \frac{g^2 f_0^2}{2\,M_0^2}\right)^{-1}, \tag{4.12}$$

$$M_\rho^2 = 2M_0^2\,. \tag{4.13}$$

[2]Notice the missprint in the original paper [15], where the authors refer M_ρ instead of M_0 in the relations for g_ρ and f at eqs. (4.11)–(4.12).

JHEP06(2007)030

The difference between the pion decay constant f and the coupling f_0 is due to the presence of $\pi - A_1$ mixing terms in the gauge chiral model lagrangian. A similar thing happens with the coupling g and the effective $\rho - \pi\pi$ parameter g_ρ. By means of eq. (4.8) one gets $\Gamma_\rho^{(0)} = g_\rho^2 M_\rho/48\pi$ and then it is not difficult to realize that the corresponding low-energy couplings in eqs. (4.9)–(4.10) exactly agree our sum-rule predictions in eqs. (3.25)–(3.26).

The parameters M_ρ, f, g_ρ are independent of the pion mass at large-N_C and, hence, the α_V corresponding to the gauge chiral model is given by

$$\alpha_V = \left[\frac{M_\rho^5}{\Gamma_\rho}\frac{d}{dm_\pi^2}\left(\frac{\Gamma_\rho}{M_\rho^3}\right)\right]_{m_\pi^2=0} = -6\,. \tag{4.14}$$

Since there are no scalars in the theory, the relation in eq. (3.27) is trivially obeyed for any value of M_ρ, g_ρ and f, and no resonance constraint is extracted.

This illustrates that renormalizability is not a necessary condition for the fulfillment of our resonance constraints. The key-point is that the amplitudes must obey a proper high energy behavior. The inspection of the $IJ = 11$ $\pi\pi$-scattering amplitude at $s \to \infty$ yields,

$$T_1^1(s) = \frac{s}{96\pi f^2}\left[1 - \frac{3\,g_\rho^2 f^2}{M_\rho^2}\right] + \mathcal{O}(s^0)\,. \tag{4.15}$$

Although one could a priori expect the presence of $\mathcal{O}(s\,m_\pi^2)$ terms, they disappear from the amplitude after precise cancelations between different contributions. The absence of these terms explains why our α_V relation in eq. (3.27) is trivially obeyed and produces no constraint on the resonance couplings. Moreover, by demanding that the $\mathcal{O}(s)$ term vanishes one gets $3g_\rho^2 f^2/M_\rho^2 = 1$, which is nothing else but the KSRF relation in eq. (3.21) in the absence of scalars. The analysis of the $IJ = 00$ and $IJ = 20$ channels gives identical results.

4.3 Minimal resonance chiral theory

In the original Resonance Chiral Theory lagrangian (RχT) proposed in ref. [17], the authors built the most general chiral invariant lagrangian that contributed at low energies to the $\mathcal{O}(p^4)$ χPT couplings. For sake of this, just operators with at most one resonance field were considered:

$$\mathcal{L}_V = \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle + \frac{iG_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu, u^\nu]\rangle\,, \tag{4.16}$$

$$\mathcal{L}_S = c_d\langle S u^\mu u_\mu\rangle + c_m\langle S\chi_+\rangle\,, \tag{4.17}$$

and the kinetic terms

$$\mathcal{L}_V^{\text{Kin}} = -\frac{1}{2}\langle\nabla^\mu V_{\mu\lambda}\nabla_\nu V^{\nu\lambda}\rangle + \frac{1}{4}M_{\text{V}}^2\langle V_{\mu\nu}V^{\mu\nu}\rangle\,, \tag{4.18}$$

$$\mathcal{L}_S^{\text{Kin}} = \frac{1}{2}\langle\nabla^\mu S\nabla_\mu S\rangle - \frac{1}{2}M_{\text{S}}^2\langle SS\rangle\,, \tag{4.19}$$

where the chiral tensors $u^\mu \sim p_\pi$, $\chi_+ \sim m_\pi^2$ and $f_+^{\mu\nu}$ containing external vector and axial-vector sources are defined in ref. [17]. The spin-1 fields are given in the antisymmetric tensor formalism. The resonance masses did not depend on the quark masses in the original approach.

The vector width was found to be

$$\Gamma_{\rm V} \,=\, \Gamma_{\rm V}^{(0)} \left[1 \,+\, \frac{m_\pi^2}{M_{\rm V}^2} \left(-6 - \frac{16 c_d c_m M_{\rm V}^2}{f^2 M_{\rm S}^2} \right) \,+\, \ldots \right] , \tag{4.20}$$

with the chiral limit of the $\rho \to \pi\pi$ width,

$$\Gamma_{\rm V}^{(0)} = \frac{G_V^2 M_{\rm V}^3}{48\pi f^4} \, . \tag{4.21}$$

The first term in the m_π^2 correction comes from the $V\pi\pi$ vertex and the width phase-space factor $\rho(M_V^2)$. The second term, proportional to $c_d c_m/M_{\rm S}^2$, comes from the pion wave function renormalization at large-N_C. It appears for $m_q \neq 0$ due to the coupling of the isosinglet resonances to the vacuum through the operator $c_m\langle S\chi_+\rangle$ [19, 20].

The corresponding scalar width is

$$\Gamma_{\rm S} \,=\, \Gamma_{\rm S}^{(0)} \left[1 \,+\, \frac{m_\pi^2}{M_{\rm S}^2} \left(-6 + \frac{4 c_m}{c_d} - \frac{16 c_d c_m}{f^2} \right) \,+\, \ldots \right] , \tag{4.22}$$

with the $\sigma \to \pi\pi$ width in the chiral limit,

$$\Gamma_{\rm S}^{(0)} = \frac{3 c_d^2 M_{\rm S}^3}{16\pi f^4} \, . \tag{4.23}$$

When we refer to σ, we denote the SU(2) singlet $\sigma = \sqrt{\frac{2}{3}} S_0 - \sqrt{\frac{1}{3}} S_8 \sim \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d)$. The first term in the m_π^2 correction is produced by the $S\pi\pi$ vertex in the $c_d\langle S u_\mu u^\mu\rangle$ operator and the width phase-space factor $\rho(M_S^2)$. The second contribution is produced by the $S\pi\pi$ vertex from the $c_m\langle S\chi_+\rangle$ operator. Finally, the last term, proportional to $c_d c_m/M_{\rm S}^2$, comes from the pion wave function renormalization and it is also utterly linked to the $c_m\langle S\chi_+\rangle$ operator.

Substituting the widths provided by the minimal RχT [17] into the modified-KSRF relation of eq. (3.21) one gets

$$1 \,=\, \frac{2 c_d^2}{f^2} \,+\, \frac{3 G_V^2}{f^2} \, . \tag{4.24}$$

Since the RχT is explicitly chiral invariant, at low energies one recovers the χPT structure independently of the value of the resonance parameters $M_{\rm V}$, $M_{\rm S}$, G_V, c_d, c_m. Eqs. (3.25) and (3.26) leads to the low energy coupling determinations

$$L_2 \,=\, 12\pi f^4 \frac{\Gamma_{\rm V}^{(0)}}{M_{\rm V}^{(0)\,5}} \,=\, \frac{G_V^2}{4 M_{\rm V}^2} \, , \tag{4.25}$$

$$L_3 \,=\, 4\pi f^4 \left(\frac{2\Gamma_{\rm S}^{(0)}}{3 M_{\rm S}^{(0)\,5}} - \frac{9\Gamma_{\rm V}^{(0)}}{M_{\rm V}^{(0)\,5}} \right) \,=\, -\frac{3 G_V^2}{4 M_{\rm V}^2} + \frac{c_d^2}{2 M_{\rm S}^2} \, , \tag{4.26}$$

in complete agreement with the expressions from the explicit integration of the heavy resonances in the RχT action [17].

The chiral corrections to the ratios $\Gamma_{\rm R}/M_{\rm R}^3$ take the form

$$\alpha_V \,=\, -6 - \frac{16 c_d c_m M_{\rm V}^2}{f^2 M_{\rm S}^2}\,, \qquad\qquad \alpha_S \,=\, -6 + \frac{4c_m}{c_d} - \frac{16 c_d c_m}{f^2}\,. \tag{4.27}$$

The substitution of these values in eq. (3.27) leads to the constraint

$$\left[1 - \frac{4c_d^2}{f^2}\right] c_m \,=\, \frac{6\,G_V^2}{f^2}\, c_m\,, \tag{4.28}$$

leading to the upper bound $G_V^2 \leq f^2/6$. This is in contradiction with the phenomenological value of the vector coupling, which is found to be $G_V^2/f^2 \sim 0.5$ [21].

It is remarkable that all the problem is originated by one single operator, $c_m\langle S\chi_+\rangle$. In the absence of this term ($c_m = 0$), one has $\alpha_S = \alpha_V = -6$ and eq. (3.27) is trivially fulfilled. This means that we cannot just add this single operator to the lagrangian. It must be accompanied by extra appropriate terms.

4.4 Extended resonance chiral theory

The study of three-point QCD Green-functions at short distances has shown that the original lagrangian is insufficient [22]. Problems have also arisen in the analysis at next-to-leading order in $1/N_C$ [23]. In general, a lagrangian made of operators including just one resonance field produces wrong growing behaviors of the amplitudes at high energies, inconsistent with perturbative QCD and the operator product expansion [24]. During recent years, different groups have worked on the development of lagrangian including operators with two and three resonance fields [22, 25, 26]. A final compilation of this operators can be found in ref. [18].

We firstly focus ourselves on the scalar sector of the theory. The relevant operators for the scalar mass and width are [18, 26]

$$\mathcal{L}_S \,=\, \lambda_6^S\langle S\{\chi_+, u^\mu u_\mu\}\rangle \,+\, \lambda_7^S\langle S u^\mu \chi_+ u_\mu\rangle\,, \tag{4.29}$$

$$\mathcal{L}_{SS} \,=\, \lambda_1^{SS}\langle SSu^\mu u_\mu\rangle \,+\, \lambda_2^{SS}\langle Su^\mu S u_\mu\rangle \,+\, \lambda_3^{SS}\langle SS\chi_+\rangle\,, \tag{4.30}$$

$$\mathcal{L}_{SSS} \,=\, \lambda_0^{SSS}\langle SSS\rangle \,+\, \lambda_1^{SSS}\langle S\nabla^\mu S\nabla_\mu S\rangle\,. \tag{4.31}$$

In the scalar sector, the presence of the operator $c_m\langle S\chi_+\rangle$ in the lagrangian induces non-zero vacuum expectation value of the isosinglet field proportional to the quark masses. For non-zero quark masses, one needs to perform the shift $S = \overline{S} + 4c_m B_0 \mathcal{M}/M_{\rm S}^{(0)\,2}$, with $M_{\rm S}^{(0)}$ the scalar mass in the chiral limit and $\mathcal{M}$ the quark mass matrix. An alternative covariant shift would be $S = \overline{S} + c_m\chi_+/M_{\rm S}^{(0)\,2}$ but the former one is more convenient for our calculation. This induces a wave-function renormalization of the pion and scalar fields, $\pi = Z_\pi^{\frac{1}{2}}\pi^r$ and $\overline{S} = Z_S^{\frac{1}{2}} S^r$, respectively.

For the large-N_C analysis of the $\pi\pi$-scattering, we can restrict ourselves to the U(2) sector of the theory and work within the isospin limit. Hence, the relevant operators for the mass and width $\Gamma[S^r \to \pi^r\pi^r]$ of the U(2)-isosinglet scalar are given up to order m_π^2 by

$$\Delta\mathcal{L} = -\frac{1}{2} M_{\rm S}^{\rm eff\ 2}\,\langle S^r S^r\rangle \,+\, c_d^{\rm eff}\,\langle S^r u^\mu u_\mu\rangle \,+\, c_m^{\rm eff}\,\langle S^r\chi_+\rangle\,, \tag{4.32}$$

with the m_π^2 dependent parameter,

$$c_d^{\rm eff} \,=\, c_d \,\left[1+\delta c_d\,\frac{m_\pi^2}{M_{\rm S}^{(0)\,2}}\right]\,, \tag{4.33}$$

given by the correction

$$\delta c_d \,=\, \frac{2M_{\rm S}^{(0)\,2}}{c_d}(2\lambda_6^S+\lambda_7^S)+\frac{4c_m}{c_d}(\lambda_1^{SS}+\lambda_2^{SS})-2\lambda_1^{SSS}c_m\,. \tag{4.34}$$

The $\mathcal{O}(m_\pi^2)$ terms in $c_m^{\rm eff}=c_m\,\left[1+\mathcal{O}(m_\pi^2)\right]$ and $M_{\rm S}^{\rm eff}=M_{\rm S}^{(0)}\,\left[1+\mathcal{O}(m_\pi^2)\right]$ are not relevant for our problem since they contribute to the ratio $\Gamma_{\rm S}/M_{\rm S}^3$ at order m_π^4. The pion decay constant up to $\mathcal{O}(m_\pi^2)$ is provided in the large-N_C limit by [17, 20]

$$f_\pi \,=\, f\,Z_\pi^{-\frac{1}{2}} \,=\, f\,\left[1\,+\,\delta f\,\frac{m_\pi^2}{M_{\rm S}^{(0)\,2}}\right]\,, \qquad\qquad \text{with}\quad \delta f\,=\,\frac{4c_dc_m}{f^2}\,. \tag{4.35}$$

In what follows, we will denote the mass $M_{\rm S}^{\rm eff}$ simply as $M_{\rm S}$, keeping $M_{\rm S}^{(0)}$ for its chiral limit.

The relevant quantities in our KSRF relations in eqs. (3.21) and (3.27) are the ratios Γ/M^3. In the scalar case, one finds

$$\begin{aligned}\frac{\Gamma_{\rm S}}{M_{\rm S}^3} \,&=\, \frac{3\,c_d^{\rm eff\,2}\,\rho(M_{\rm S}^2)}{16\pi\,f_\pi^4}\,\left[1\,+\,\frac{4m_\pi^2}{M_{\rm S}^{(0)\,2}}\left(\frac{c_m}{c_d}-1\right)\right] \\ &=\, \frac{3\,c_d^{\rm eff\,2}}{16\pi\,f_\pi^4}\,\left[1\,+\,\frac{m_\pi^2}{M_{\rm S}^{(0)\,2}}\left(\frac{4c_m}{c_d}-6\right)\,+\,\mathcal{O}(m_\pi^4)\right]\,.\end{aligned} \tag{4.36}$$

The global coefficients provides that chiral limit of $\Gamma_{\rm S}/M_{\rm S}^3$ found in the previous section. The chiral corrections are there given in terms of the combination of couplings

$$(6+\alpha_S)\,=\,6\,+\,\left[\frac{M_S^5}{\Gamma_S}\,\frac{d}{dm_\pi^2}\left(\frac{\Gamma_S}{M_S^3}\right)\right]_{m_\pi^2=0} \,=\, 2\,\delta c_d\,-\,4\,\delta f\,+\,\frac{f^2}{c_d^2}\,\delta f\,. \tag{4.37}$$

It is possible to carry a similar analysis for the vector meson, expressing the $V^r\,\to\,\pi^r\pi^r$ in terms of the effective parameter $G_V^{\rm eff}$. Though the explicit form of $G_V^{\rm eff}=G_V\left[1+\delta G_V\frac{m_\pi^2}{M_{\rm V}^{(0)\,2}}\right]$ is not given in this paper, we can write:

$$\begin{aligned}\frac{\Gamma_{\rm V}}{M_{\rm V}^3} \,&=\, \frac{G_V^{\rm eff\,2}\,\rho(M_{\rm V}^2)^3}{48\pi f_\pi^4} \\ &=\, \frac{G_V^{\rm eff\,2}}{48\pi f_\pi^4}\,\left[1\,-\,\frac{6\,m_\pi^2}{M_{\rm V}^{(0)\,2}}\,+\,\mathcal{O}(m_\pi^4)\right]\,,\end{aligned} \tag{4.38}$$

which gives $(6+\alpha_V)=2\,\delta G_V-4\,\delta f\,\frac{M_V^{(0)\,2}}{M_S^{(0)\,2}}$.

Gathering all the information from RχT in eqs. (4.36)–(4.38), one gets for the modified-KSRF relation in eq. (3.21) and the new $\alpha_V - \alpha_S$ relation in eq. (3.27) the result

$$\frac{3\,G_V^2}{f^4} + \frac{2\,c_d^2}{f^4} = \frac{1}{f^2}\,, \qquad (4.39)$$

$$\frac{3\,G_V^2}{f^4}\left[\frac{2\,\delta G_V}{M_V^{(0)\,2}} - \frac{4\,\delta f}{M_S^{(0)\,2}}\right] + \left[\frac{2c_d^2}{f^4}\,\frac{(2\delta c_d - 4\delta f)}{M_S^{(0)\,2}} + \frac{1}{f^2}\,\frac{2\,\delta f}{M_S^{(0)\,2}}\right] = 0\,, \qquad (4.40)$$

where a global factor $1/16\pi$ has been simplified with respect to eqs. (3.21) and (3.27). It is not difficult to put the two former equations together into the single relation

$$\frac{3\,G_V^{\rm eff\;2}}{f_\pi^4} + \frac{2\,c_d^{\rm eff\;2}}{f_\pi^4} = \frac{1}{f_\pi^2}\,. \qquad (4.41)$$

The leading order in its m_π^2 expansion provides eq. (4.39) and its $\mathcal{O}(m_\pi^2)$ term produces eq. (4.40). A last simplification of a global factor $1/f_\pi^2$ is left for the reader. It is remarkable that both resonance constraints are actually governed in RχT by the ratios $\frac{c_d^{\rm eff\;2}}{f_\pi^2}$ and $\frac{G_V^{\rm eff\;2}}{f_\pi^2}$.

Once again, the analysis of the $\pi\pi$-scattering amplitude at high energies allows a better understanding of our sum-rule result. We find,

$$T(s)_1^1 = \frac{s}{96\pi f_\pi^2}\left[1 - \frac{3\,G_V^{\rm eff\;2}}{f_\pi^2} - \frac{2\,c_d^{\rm eff\;2}}{f_\pi^2}\right] + \mathcal{O}(s^0)\,. \qquad (4.42)$$

Identical results are found for the $IJ = 00$ and $IJ = 20$ channels.

5. The scalar resonance at $N_C = 3$ and $N_C \to \infty$

Historically, the understanding on the scalar sector is much less clear than the vector sector. In ref. [27] it is demonstrated that (when $N_C = 3$) a light and broad scalar resonance (the σ meson) dominates at low energies in the IJ=00 channel and takes an essential role to adjust chiral perturbation theory to experiments. The pole location is estimated in [9] using the dispersion representation eq. (2.1), which are in good agreement with the more rigorous Roy equation analysis [28]. Under this situation it is worthwhile to investigate the role of these light and broad resonances.

It is not clear, however, what is the nature of this σ meson and different opinions exist on its large N_c behavior. The σ meson may even be considered as a dynamically generated resonance and decouples in some way from low energy physics when N_C is large [29]. For example, the K matrix unitarization of the current algebra term yields a σ pole in the chiral limit with the following pole location:

$$z_\sigma \simeq 16 i\pi f_\pi^2\,. \qquad (5.1)$$

This 'current algebra σ' maintains an unusual property: It flies away on the complex s plane meanwhile it contributes to the *r.h.s.* of eq. (3.21) in the large N_C limit. Nonetheless, such a pole does not contribute to the sum rule for L_3, i.e., eq. (3.26). Indeed the existence of poles which moves to ∞ can not be excluded using pure N_c counting rule. However in the s channel such a pole contributes, in the chiral limit, a term

$$T^{\rm sR}(s) = \frac{\frac{s}{16\pi f^2}}{1 - i\rho\frac{s}{16\pi f^2}} \tag{5.2}$$

to the *r.h.s.* of eq. (2.8), according to eqs. (2.3) and (2.4). However, unlike the ordinary narrow resonances, crossing symmetry is not fulfilled. Beside this, the 'current algebra σ' in eq. (5.2) contributes $1/16\pi f^2$ to the r.h.s. of eq. (3.21). This is misleading since the KSRF relation eq. (3.21) tells where the factor $1/16\pi f^2$ comes from. Furthermore, the unitarization of the current algebra amplitude produces unphysical poles $z_\rho = 96 i\pi f^2$ in the second Riemann sheet and $z_{(20)} = 32 i\pi f^2$ in the first Riemann sheet. This leads to an incorrect interpretation of the KSRF relation.

It is important to notice that the behavior of σ-meson must be totally different in the case when $N_c \to \infty$. It is noticed that the N_c dependent pole trajectory for σ behaves very differently from that of ρ [29]. This phenomenon is re-investigated in ref. [30]. It is found that, even though the σ pole trajectory is bent from the expected large-N_C behaviour, it can finally fall down to the real axis at $N_C \to \infty$ and, hence, be relevant at large-N_C. It is argued in ref. [31] that the bent structure of the σ pole trajectory itself is not sufficient to demonstrate that the σ pole is dynamically generated. Although these investigations are based on models and other assumptions, they show that this alternative scenario should not be ruled out.

We want to finish with a numerical analysis of eqs. (3.21), (3.25) and (3.26), where we will consider the inputs $f = 88\,\text{MeV}$, $M_\rho = 770\,\text{MeV}$, $\Gamma_\rho = 146\,\text{MeV}$. Since we assume that the scalar becomes a narrow-width state at $N_C \to \infty$, the values of M_σ and Γ_σ should be different from their corresponding values at $N_C = 3$. Here we adopt a rather exaggeratory value of the scalar parameters, $M_\sigma = 700\,\text{MeV}$ and $\Gamma_\sigma = 500\,\text{MeV}$. For the r.h.s. of the modified-KSRF relation in eq. (3.21), one has (in units of GeV^{-2})

$$\frac{9\,\Gamma_V^{(0)}}{M_V^{(0)\,3}} + \frac{2\,\Gamma_S^{(0)}}{3\,M_S^{(0)\,3}} \simeq 2.9 + 1.0\,, \tag{5.3}$$

where the first term on the right-hand side comes from the vector contribution and the second one from the scalar. From the modified-KSRF relation, one would expect their sum to be equal to $1/16\pi f^2 \simeq 2.6\,\text{GeV}^{-2}$. Although these large-$N_C$ estimates are rough, they suggest that there is almost no room for the scalar contribution to the *r.h.s* of eq. (3.21). Thus, in the picture suggested in ref. [32], the bare σ mass turns out to be of the order of $M_\sigma \sim 1\,\text{GeV}$, resulting the scalar contribution indeed suppressed by the large mass and becoming the modified-KSRF relation insensitive to the value of Γ_σ.

Our numerical prediction for L_2 and L_3 at large-N_C is

$$10^3 \cdot L_2 \simeq 1.2\,, \tag{5.4}$$

$$10^3 \cdot L_3 \simeq -3.7 + 1.5\,, \tag{5.5}$$

where the first contribution to L_3 comes from the vector meson and the last one from the scalar. This can be compared to the one-loop experimental determination, $10^3\,L_2^r =$

1.35 ± 0.3, $10^3\, L_3^r = -3.5 \pm 1.1$ [33], and to Bijnens' $\mathcal{O}(p^6)$ result, $10^3\, L_2^r = 0.73 \pm 0.12$, $10^3\, L_3^r = -2.35 \pm 0.37$ [34]. It is possible to isolate the scalar resonance contribution to the LECs by considering an appropriate combination of eqs. (3.25) and (3.26):

$$L_3 + 3L_2 = \frac{8\pi f^4\, \Gamma_\sigma}{3M_\sigma^5} > 0\,, \tag{5.6}$$

which, for our input values $M_\sigma = 700\,\text{MeV}$, $\Gamma_\sigma = 500\,\text{MeV}$, yields

$$L_3 + 3L_2 \simeq 1.5 \cdot 10^{-3}\,. \tag{5.7}$$

The experimental determinations for L_2 and L_3 in χPT provide the upper bound $L_3^r + 3L_2^r \lesssim 1.9 \cdot 10^{-3}$ at one loop [33] and $L_3^r + 3L_2^r \lesssim 0.36 \cdot 10^{-3}$ at $\mathcal{O}(p^6)$ [34]. This indicates that, at large-N_C, either Γ_σ is small or M_σ becomes large. For example, for $\Gamma_\sigma = 500\,\text{MeV}$, the smallest value for the mass is $M_\sigma \simeq 670\,\text{MeV}$ if the one-loop upper bound is assumed, and $M_\sigma \simeq 930\,\text{MeV}$ if we take the $\mathcal{O}(p^6)$ result. Nevertheless, it is important to recall that experimental determinations of the LECs differ from the corresponding values at large-N_C due to subleading corrections in $1/N_C$ [23], so one should be cautious about these bounds.

In any case, the safe conclusion from eq. (3.21) is that the scalar meson takes a numerically minor role in the KSRF relation when N_C is large. The situation can be quite different in the $N_C = 3$ case. For instance, the present work shows that the $IJ = 00$ scattering length is dominated by the crossed-channel ρ exchange at large-N_C. However, the phenomenological analysis of the $IJ = 00$ experimental data is found to be dominated by the s-channel scalar contribution [9].

6. Discussions and conclusions

In this paper we started from a variation of partial wave dispersion relation (the PKU form) and demonstrated that it is reduced to the standard once subtracted partial wave dispersion relation (PWDR) in the narrow width approximation or in the leading order of $1/N_c$ expansion. Matching the resonance contribution calculated from PWDR to the low energy chiral amplitudes up to $\mathcal{O}(p^4)$ leads to a set of resonance sum rules. They include the KSRF relation, two sum rules for the low energy constants L_2, L_3 and a new relation between resonance couplings, eq. (3.27).

We made a rather detailed examination of the new relation in various resonance chiral lagrangians and found that it is not always trivially fulfilled. Hence it provides a useful novel constraint for the construction of the hadronic action. The origin of this constraint is understood: It comes from the requirement of chiral symmetry and a proper high energy behavior of the scattering amplitude. We start from an S matrix theory point of view, which is crucial to provide a rigorous and systematic way to derive the sum rules, independently of the realization of the resonance lagrangian. Our investigation provides a clearer understanding to the KSRF relation and generalizes it beyond the leading chiral order. We also discussed the N_c property of the σ meson and conclude that, unlike the case when $N_c = 3$, it takes a numerically negligible role when $N_c \to \infty$.

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References

[1] H. Georgi, *Effective field theory*, *Ann. Rev. Nucl. Part. Sci.* **43** (1993) 209;
A.V. Manohar, *Effective field theories*, hep-ph/9606222;
A. Pich, *Effective field theory*, hep-ph/9806303.

[2] S. Weinberg, *Phenomenological Lagrangians*, *Physica* **96** (1979) 327.

[3] J. Gasser and H. Leutwyler, *Chiral perturbation theory to one loop*, *Ann. Phys. (NY)* **158** (1984) 142.

[4] J. Gasser and H. Leutwyler, *Chiral perturbation theory: expansions in the mass of the strange quark*, *Nucl. Phys.* **B 250** (1985) 465.

[5] Z. Xiao and H.Q. Zheng, *Analyticity and the n(c) counting rule of S matrix poles*, *Mod. Phys. Lett.* **A 22** (2007) 55 [hep-ph/0502199].

[6] H.Q. Zheng et al., *The κ resonance in S wave πK scatterings*, *Nucl. Phys.* **A 733** (2004) 235 [hep-ph/0310293];
Z.Y. Zhou and H.Q. Zheng, *An improved study of the kappa resonance and the non-exotic s wave πK scatterings up to $\sqrt{s} = 2.1$ GeV of LASS data*, *Nucl. Phys.* **A 775** (2006) 212.

[7] J.-y. He, Z. Xiao and H.Q. Zheng, *The constraints of unitarity on pi pi scattering dispersion relations*, *Phys. Lett.* **B 536** (2002) 59 [*Erratum ibid.* **549** (2002) 362] [hep-ph/0201257].

[8] Q. Ang et al., *A critical examination to the unitarized $\pi\pi$ scattering chiral amplitudes*, *Commun. Theor. Phys.* **36** (2001) 563 [hep-ph/0109012].

[9] Z.Y. Zhou et al., *The pole structure of the unitary, crossing symmetric low energy $\pi\pi$ scattering amplitudes*, *JHEP* **02** (2005) 043 [hep-ph/0406271].

[10] B.R. Martin, D. Morgan and G. Shaw, *Pion pion interactions in particle physics*, Academic Press, London, U.K. (1976).

[11] K. Kawarabayashi and M. Suzuki, *Partially conserved axial vector current and the decays of vector mesons*, *Phys. Rev. Lett.* **16** (1966) 255;
Riazuddin and Fayazuddin, *Algebra of current components and decay widths of ρ and K^* mesons*, *Phys. Rev.* **147** (1966) 1071.

[12] J.L. Basdevant and J. Zinn-Justin, *Yang-Mills fields and the $\pi\pi$ interaction*, *Phys. Rev.* **D 3** (1971) 1865;
See also S. Rudaz, *Comment on vector-meson-dominance and the $\gamma \to 3\pi$ vertex in the low-energy limit*, *Phys. Rev.* **D 10** (1974) 3857;
G. Kramer and W.F. Palmer, *Chiral anomalies and vector meson dominance*, *Phys. Rev.* **D 36** (1987) 154.

[13] K. Igi and K.-i. Hikasa, *Another look at pi pi scattering in the scalar channel*, *Phys. Rev.* **D 59** (1999) 034005 [hep-ph/9807326]; *W W scattering with scalar and vector resonances*, *Phys. Rev.* **D 48** (1993) 3055.

[14] T.N. Pham and T.N. Truong, *Evaluation of the derivative quartic terms of the meson chiral lagrangian from forward dispersion relation*, *Phys. Rev.* **D 31** (1985) 3027.

[15] J.F. Donoghue, C. Ramirez and G. Valencia, *The spectrum of QCD and chiral lagrangians of the strong and weak interactions*, *Phys. Rev.* **D 39** (1989) 1947.

[16] M. Bando, T. Kugo, S. Uehara, K. Yamawaki and T. Yanagida, *Is ρ meson a dynamical gauge boson of hidden local symmetry?*, *Phys. Rev. Lett.* **54** (1985) 1215;
M. Harada and K. Yamawaki, *Hidden local symmetry at one loop*, *Phys. Lett.* **B 297** (1992) 151 [hep-ph/9210208].

[17] G. Ecker, J. Gasser, A. Pich and E. de Rafael, *The role of resonances in chiral perturbation theory*, *Nucl. Phys.* **B 321** (1989) 311.

[18] V. Cirigliano et al., *Towards a consistent estimate of the chiral low-energy constants*, *Nucl. Phys.* **B 753** (2006) 139 [hep-ph/0603205].

[19] M. Jamin, J.A. Oller and A. Pich, *Strangeness-changing scalar form factors*, *Nucl. Phys.* **B 622** (2002) 279 [hep-ph/0110193].

[20] J.J. Sanz-Cillero, *Pion and kaon decay constants: lattice vs. resonance chiral theory*, *Phys. Rev.* **D 70** (2004) 094033 [hep-ph/0408080].

[21] G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, *Chiral lagrangians for massive spin 1 fields*, *Phys. Lett.* **B 223** (1989) 425.

[22] B. Moussallam, *Chiral sum rules for parameters of the order six lagrangian in the W-Z sector and application to pi0, eta, eta-prime decays*, *Phys. Rev.* **D 51** (1995) 4939 [hep-ph/9407402]; *A sum rule approach to the violation of Dashen's theorem*, *Nucl. Phys.* **B 504** (1997) 381 [hep-ph/9701400];
M. Knecht and A. Nyffeler, *Resonance estimates of $O(p^6)$ low-energy constants and QCD short-distance constraints*, *Eur. Phys. J.* **C 21** (2001) 659 [hep-ph/0106034];
P.D. Ruiz-Femenia, A. Pich and J. Portoles, *Odd-intrinsic-parity processes within the resonance effective theory of QCD*, *JHEP* **07** (2003) 003 [hep-ph/0306157];
V. Cirigliano, G. Ecker, M. Eidemuller, A. Pich and J. Portoles, *The Green function in the resonance region*, *Phys. Lett.* **B 596** (2004) 96 [hep-ph/0404004];
V. Cirigliano et al., *The Green function and* SU(3) *breaking in $K(l3)$ decays*, *JHEP* **04** (2005) 006 [hep-ph/0503108].

[23] O. Cata and S. Peris, *An example of resonance saturation at one loop*, *Phys. Rev.* **D 65** (2002) 056014 [hep-ph/0107062];
I. Rosell, J.J. Sanz-Cillero and A. Pich, *Quantum loops in the resonance chiral theory: the vector form factor*, *JHEP* **08** (2004) 042 [hep-ph/0407240];
I. Rosell, P. Ruiz-Femenia and J. Portoles, *One-loop renormalization of resonance chiral theory: scalar and pseudoscalar resonances*, *JHEP* **12** (2005) 020 [hep-ph/0510041];
I. Rosell, J.J. Sanz-Cillero and A. Pich, *Towards a determination of the chiral couplings at NLO in $1/N_c$: $L_8(r_\mu)$*, *JHEP* **01** (2007) 039 [hep-ph/0610290];
J.J. Sanz-Cillero, *Resonance form-factors: $L(8)$ determination at next-to-leading order in $1/N_c$*, hep-ph/0610304.

[24] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *QCD and resonance physics. Sum rules*, *Nucl. Phys.* **B 147** (1979) 385.

[25] V. Cirigliano, G. Ecker, H. Neufeld and A. Pich, *Meson resonances, large-N_c and chiral symmetry*, *JHEP* **06** (2003) 012 [hep-ph/0305311].

[26] I.Rosell, J.J. Sanz-Cillero and A. Pich, in preparation.

[27] Z. Xiao and H.Q. Zheng, *Left hand singularities, hadron form factors and the properties of the sigma meson*, *Nucl. Phys.* **A 695** (2001) 273 [hep-ph/0011260]; *Further remarks on $\pi\pi$ scattering dispersion relations*, *Chin. Phys. Lett.* **20** (2003) 342 [hep-ph/0206247].

[28] I. Caprini, G. Colangelo and H. Leutwyler, *Mass and width of the lowest resonance in QCD*, *Phys. Rev. Lett.* **96** (2006) 132001 [hep-ph/0512364].

[29] J.R. Pelaez, *On the nature of light scalar mesons from their large-N_c behavior*, *Phys. Rev. Lett.* **92** (2004) 102001 [hep-ph/0309292].

[30] Z.X. Sun, L.Y. Xiao, Z. Xiao and H.Q. Zheng, *Model dependent analyses on the N_c dependence of the sigma pole trajectory*, *Mod. Phys. Lett.* **A 22** (2007) 711 [hep-ph/0503195] talk presented at *Conference on non-perturbative quantum field theory: lattice and beyond*, Guangzhou, China, (2004).

[31] Z.-H. Guo, L.Y. Xiao and H.Q. Zheng, *Is the $F_0(600)$ meson a dynamically generated resonance? A lesson learned from the $O(N)$ model and beyond*, hep-ph/0610434.

[32] L.Y. Xiao, H.Q. Zheng and Z.Y. Zhou, *Lightest scalars as chiral partners of the Nambu-Goldstone bosons*, hep-ph/0609009, talk given at *13th International Conference in QCD* (QCD 06), Montpellier, France, 3-7 Jul (2006).

[33] J. Bijnens, G. Colangelo and J. Gasser, *$K(l4)$ decays beyond one loop*, *Nucl. Phys.* **B 427** (1994) 427 [hep-ph/9403390].

[34] G. Amoros, J. Bijnens and P. Talavera, *QCD isospin breaking in meson masses, decay constants and quark mass ratios*, *Nucl. Phys.* **B 602** (2001) 87 [hep-ph/0101127].